A Semidefinite Relaxation Approach to the Geolocation of Two Unknown Co-channel Emitters by a Cluster of Formation-Flying Satellites Using Both TDOA and FDOA Measurements

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Motivation

- A few unknown emitters fall into the same frequency channel of satellite receivers in satellite formation flying geolocation systems;

- A new geolocation algorithm is required.

Satellite Formation Flying Geolocation System

A cluster of satellites fly with a particularly designed configuration and each satellite flies in its own orbit: [With the use of Satellite Tool Kit (STK)]

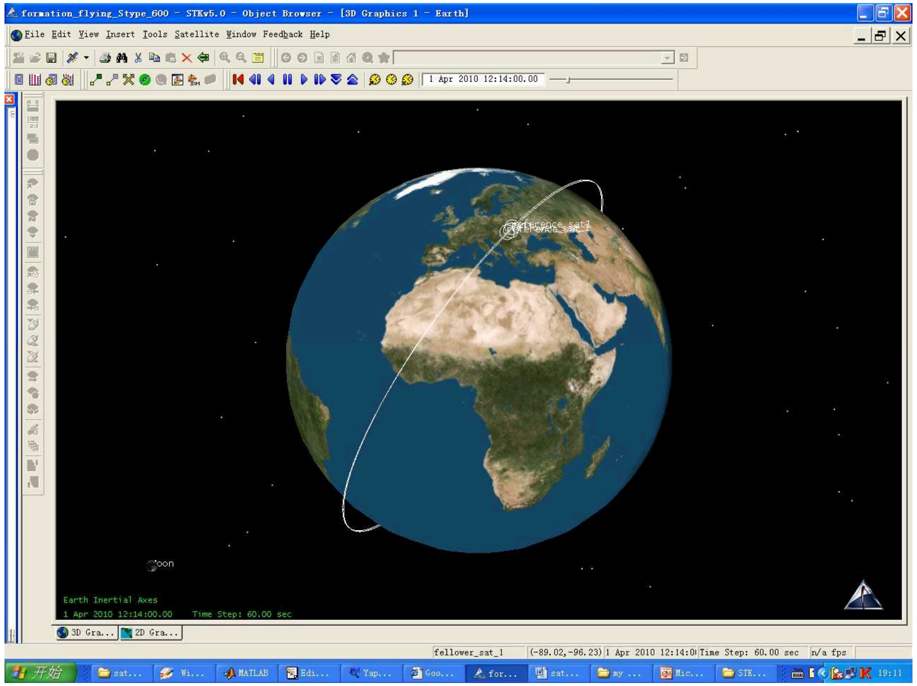


Figure 1: Three satellites formation flying Geolocation System
(overall view)

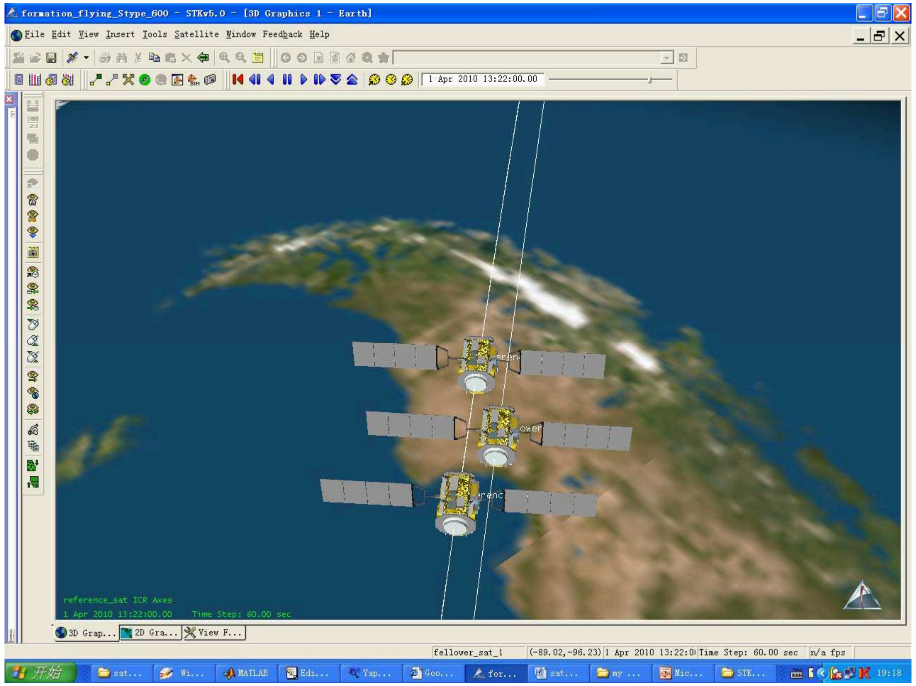
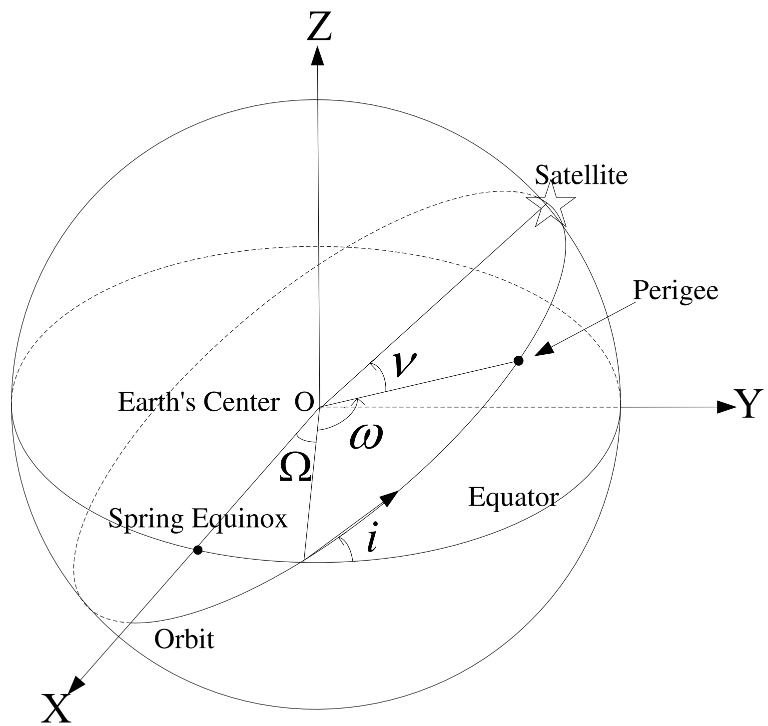


Figure 2: Three satellites formation flying Geolocation System
(local enlarge)

Satellite Flying with Six Orbital Parameters

The main two elements define the shape and size of the ellipse:
① a: Semi-major axis of ellipse orbit;
② e: Orbital eccentricity, describing how flattened it is compared with a circle;

Two elements define the orientation of the orbital plane in which the ellipse is embedded:
③ Ω: Longitude of the ascending node, it is the angle from the origin of longitude to the direction of the ascending node, which specify the orbit in space;



④ i: Inclination, it is the angular distance of the orbital plane from the plane of equator;

⑤ ω: Argument of periapsis, it defines the orientation of the ellipse;
⑥ ν: Mean anomaly, it defines the position of the satellite along the ellipse at a specific time.

Flying Configuration Design by an Formation algorithm

Two typical configurations are used: ”**wheel type**” and ”**pendulum type**” by STK.

① **wheel type**:
Reference satellite is running on an elliptical orbit, and the followers are flying around it (relative) in the same plane. Each satellite has its own elliptical orbit.

$$\begin{cases} a_k = a, \\ e_k = e, \\ i_k = i + e \cdot \eta \cdot \sin(\upsilon_k), \\ \Omega_k = \Omega - e \cdot \eta \cdot \cos(\upsilon_k) \cdot \csc(i), \\ \omega_k = \varphi_k + e \cdot \eta \cdot \cos(\upsilon_k) \cdot \cot(i), \\ M_{k0} = -\varphi_k. \end{cases}$$

where, $(\eta, \upsilon_k, \varphi_k)$ are the parameters of wheel type formation figuration;

② **pendulum type**:
Reference satellite and followers are all running on circular orbits, and each follower is simple harmonic motion relative to reference satellite in the same plane.

$$\begin{cases} a_k = a, \\ e_k = 0, \\ i_k = i + \frac{\Delta Z_k \cdot \cos(\varphi_k)}{a}, \\ \Omega_k = \Omega + \frac{\Delta Z_k \cdot \sin(\varphi_k)}{a \cdot \sin(i)}, \\ \omega_k = \varphi_k + \frac{\Delta X_k}{a} - \frac{\Delta Z_k \cdot \sin(\varphi_k) \cdot \cot(i)}{a}, \\ M_{k0} = -\varphi_k \end{cases}$$

where, $(\Delta X_k, \Delta Z_k)$ are the parameters of pendulum type formation figuration.

Glossary

ECI	- Earth-centered Inertial: to show the relative motion bt. sat. and the earth [20]
ECEF	- Earth-centered (Earth) Fixed: to geolocate the target on the earth [20]
WGS 84	- World Geodetic Systems 1984 (latitude, longitude, altitude) [20]
STK	- Satellite Tool Kit
Geolocation	- localization of an emitter on the surface of the earth [4]
TDOA/FDOA	- Time Difference of Arrival / Frequency Difference of Arrival
SDR	- Semidefinite Relaxation

[20] A. C. Long, J.O. Cappellari, C. E. Velez, and A.J. Fuchs, Goddard trajectory determination system (GTDS): Mathematical Theory (Revision 1), National Aeronautics and Space Administration/Goddard Space Flight Center, 1989.

[4] Darko Musicki, Regina Kaune, and Wolfgang Koch, “Mobile emitter geolocation and tracking using TDOA and FDOA measurements”, *IEEE Trans. Signal Process.*, vol.58, no.3, pp.1863-1874, Mar. 2010.

TDOA/FDOA Measurements for Multiple Cochannel Emitters

TDOA Measurements:

$$\tau_{ij}^{(k)} = \frac{1}{c}\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\| - \frac{1}{c}\left\|\boldsymbol{x}_k - \boldsymbol{s}_j\right\| + n_{ij}^{(k)} \tag{1}$$

FDOA Measurements:

$$\nu_{ij}^{(k)} = \frac{(\boldsymbol{s}_i - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_i}{c\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|} - \frac{(\boldsymbol{s}_j - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_j}{c\left\|\boldsymbol{x}_k - \boldsymbol{s}_j\right\|} + v_{ij}^{(k)} \tag{2}$$

$\boldsymbol{x}_k \in \mathcal{R}^3$: location of the k-th unknown emitter to be estimated,
$\boldsymbol{s}_j \in \mathcal{R}^3$: location of the *j*-th satellite,
$\dot{\boldsymbol{s}}_j \in \mathcal{R}^3$: veolocity of the *j*-th satellite,
$n_{ij}^{(k)}$ and $v_{ij}^{(k)}$: TDOA and FDOA measurement noise,
$i, j \in \mathcal{I} \triangleq \{1, 2, \cdots, M\}$, M is the number of the synchronized satellites in the cluster,
$k \in \mathcal{K} \triangleq \{1, 2, \cdots, K\}$, K is the number of unknown emitters on the earth.

Formulation of the Geolocation Problem

Since $\boldsymbol{\tau}_{ij} = [\tau_{ij}^{(1)}, \ldots, \tau_{ij}^{(K)}]^T$, $\boldsymbol{\nu}_{ij} = [\nu_{ij}^{(1)}, \ldots, \nu_{ij}^{(K)}]^T$ are obtained in ascending or descending order, the geolocation becomes the mixed integer nonlinear optimization problem:

$$
\begin{aligned}
\min_{\substack{\boldsymbol{x}_k, \boldsymbol{t}_i, \boldsymbol{f}_i, \boldsymbol{P}^{(ij)}, \boldsymbol{Q}^{(ij)} \\ i,j\in\mathcal{I},\, i>j,\, k\in\mathcal{K}}} \quad & \frac{1}{\sigma_T^2} \sum_{\substack{i,j\in\mathcal{I} \\ i>j}} \left\|\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{P}^{(ij)}\boldsymbol{\tau}_{ij}\right\|^2 + \frac{1}{\sigma_F^2} \sum_{\substack{i,j\in\mathcal{I} \\ i>j}} \left\|\boldsymbol{f}_i - \boldsymbol{f}_j - \boldsymbol{Q}^{(ij)}\boldsymbol{\nu}_{ij}\right\|^2 \\
\text{s.t.} \quad & t_i^{(k)} = \frac{1}{c}\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|, \; i \in \mathcal{I}, \; k \in \mathcal{K}, \\
& f_i^{(k)} = \frac{(\boldsymbol{s}_i - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_i}{c\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|}, \; i \in \mathcal{I}, \; k \in \mathcal{K}, \\
& \boldsymbol{t}_i = \left[t_i^{(1)}, \ldots, t_i^{(K)}\right]^T, \; i \in \mathcal{I}, \\
& \boldsymbol{f}_i = \left[f_i^{(1)}, \ldots, f_i^{(K)}\right]^T, \; i \in \mathcal{I}, \\
& \boldsymbol{P}^{(ij)}, \; \boldsymbol{Q}^{(ij)} \in \Pi_K, \; i, j \in \mathcal{I}, \; i > j.
\end{aligned}
\tag{3}
$$

where Π_K is the set of $K \times K$ permutation matrices.

Weighted Bipartite Matching

When $\boldsymbol{x}_k$ is given, we can obtain:

$$t_i^{(k)} = \frac{1}{c}\big\|\boldsymbol{x}_k - \boldsymbol{s}_i\big\|, \qquad f_i^{(k)} = \frac{(\boldsymbol{s}_i - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_i}{c\big\|\boldsymbol{x}_k - \boldsymbol{s}_i\big\|},$$

$$W_{kl}^{(ij)} = (t_i^{(k)} - t_j^{(k)} - \tau_{(ij)}^{(l)})^2, \qquad V_{kl}^{(ij)} = (f_i^{(k)} - f_j^{(k)} - \nu_{(ij)}^{(l)})^2,$$

$$\boldsymbol{W}^{(ij)} = \{W_{kl}^{(ij)},\ k, l \in \mathcal{K}\}, \qquad \boldsymbol{V}^{(ij)} = \{V_{kl}^{(ij)},\ k, l \in \mathcal{K}\}.$$

where $i, j \in \mathcal{I},\ \ k, l \in \mathcal{K}$. The geolocation problem (3) becomes the standard weighted complete bipartite matching problem under the graphs $G^{(ij)} = (V, E, \boldsymbol{W}^{(ij)})$ and $\tilde{G}^{(ij)} = (V, E, \boldsymbol{V}^{(ij)})$ with bipartition $V = V_1 \cup V_2$ and $V_1 = V_2 = \mathcal{K}$:

$$\min_{\substack{\boldsymbol{P}^{(ij)}, \boldsymbol{Q}^{(ij)} \\ i,j \in \mathcal{I}, j>i}} \frac{1}{\sigma_T^2} \sum_{\substack{i,j \in \mathcal{I} \\ j>i}} \operatorname{tr}\left\{\boldsymbol{W}^{(ij)} \boldsymbol{P}^{(ij)}\right\} + \frac{1}{\sigma_F^2} \sum_{\substack{i,j \in \mathcal{I} \\ j>i}} \operatorname{tr}\left\{\boldsymbol{V}^{(ij)} \boldsymbol{Q}^{(ij)}\right\}. \tag{4}$$

[18] D. B. West, *Introduction to Graph Theory*, Prentice Hall, second edition, 2001.

Our Goal

- Using Semidefinite Relaxation (SDR) to solve this mixed integer nonlinear geolocation problem;

- Propose a geolocation algorithm.

Some Notations

Let

$$
\begin{aligned}
\tilde{\boldsymbol{t}} &= \left[(\boldsymbol{t}_1-\boldsymbol{t}_2)^T,\ldots,(\boldsymbol{t}_1-\boldsymbol{t}_M)^T,\ldots,(\boldsymbol{t}_{M-1}-\boldsymbol{t}_M)^T\right]^T = \bar{\boldsymbol{G}}\boldsymbol{t},\\
\tilde{\boldsymbol{f}} &= \left[(\boldsymbol{f}_1-\boldsymbol{f}_2)^T,\ldots,(\boldsymbol{f}_1-\boldsymbol{f}_M)^T,\ldots,(\boldsymbol{f}_{M-1}-\boldsymbol{f}_M)^T\right]^T = \bar{\boldsymbol{G}}\boldsymbol{f},\\
\boldsymbol{t} &= \left[\boldsymbol{t}_1^T,\ldots,\boldsymbol{t}_M^T\right]^T, \quad \boldsymbol{f} = \left[\boldsymbol{f}_1^T,\ldots,\boldsymbol{f}_M^T\right]^T,\\
\boldsymbol{P} &= \mathrm{blkdiag}\left(\boldsymbol{P}^{(12)},\ldots,\boldsymbol{P}^{(1M)},\ldots,\boldsymbol{P}^{(M-1,M)}\right),\\
\boldsymbol{Q} &= \mathrm{blkdiag}\left(\boldsymbol{Q}^{(12)},\ldots,\boldsymbol{Q}^{(1M)},\ldots,\boldsymbol{Q}^{(M-1,M)}\right),\\
\boldsymbol{\tau} &= \left[\boldsymbol{\tau}_{(12)}^T,\ldots,\boldsymbol{\tau}_{(1M)}^T,\ldots,\boldsymbol{\tau}_{(M-1,M)}^T\right]^T,\\
\boldsymbol{\nu} &= \left[\boldsymbol{\nu}_{(12)}^T,\ldots,\boldsymbol{\nu}_{(1M)}^T,\ldots,\boldsymbol{\nu}_{(M-1,M)}^T\right]^T,
\end{aligned}
$$

where $\bar{\boldsymbol{G}} = \boldsymbol{G} \otimes \boldsymbol{I}_K$ is a $KM(M-1)/2 \times KM$ matrix, $\boldsymbol{I}_K$ denotes the $K \times K$ identity matrix. And $\boldsymbol{G}$ is of the form:

$$\boldsymbol{G} = \begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -1 \end{bmatrix}$$

Using the above notations, the objective function of Problem (3) can be denoted by

$$\begin{aligned} \theta &= \frac{1}{\sigma_T^2}\|\tilde{\boldsymbol{t}} - \boldsymbol{P}\boldsymbol{\tau}\|^2 + \frac{1}{\sigma_F^2}\|\tilde{\boldsymbol{f}} - \boldsymbol{Q}\boldsymbol{\nu}\|^2 \\ &= \frac{1}{\sigma_T^2}\boldsymbol{t}^T\bar{\boldsymbol{G}}^T\bar{\boldsymbol{G}}\boldsymbol{t} + \frac{1}{\sigma_T^2}\|\boldsymbol{P}\boldsymbol{\tau}\|^2 - \frac{2}{\sigma_T^2}(\boldsymbol{P}\boldsymbol{\tau})^T\bar{\boldsymbol{G}}\boldsymbol{t} \\ &\quad + \frac{1}{\sigma_F^2}\boldsymbol{f}^T\bar{\boldsymbol{G}}^T\bar{\boldsymbol{G}}\boldsymbol{f} + \frac{1}{\sigma_F^2}\|\boldsymbol{Q}\boldsymbol{\nu}\|^2 - \frac{2}{\sigma_F^2}(\boldsymbol{Q}\boldsymbol{\nu})^T\bar{\boldsymbol{G}}\boldsymbol{f}. \end{aligned} \tag{5}$$

Permutation Matrices for Two Cochannel Emitters

Since each element of a permutation matrix takes values in $\{0, 1\}$ and each row and each column sums to one, for the case where $K = 2$, $\boldsymbol{P}^{(ij)}$ and $\boldsymbol{Q}^{(ij)}$ can be expressed as

$$\boldsymbol{P}^{(ij)} = \begin{bmatrix} 1 - p^{(ij)} & p^{(ij)} \\ p^{(ij)} & 1 - p^{(ij)} \end{bmatrix}, \tag{6}$$

$$\boldsymbol{Q}^{(ij)} = \begin{bmatrix} 1 - q^{(ij)} & q^{(ij)} \\ q^{(ij)} & 1 - q^{(ij)} \end{bmatrix}, \tag{7}$$

where $p^{(ij)}, q^{(ij)} \in \{0, 1\}$. In particular, there exist matrices $\boldsymbol{B}_p$ and $\boldsymbol{B}_q$ such that

$$\boldsymbol{P}\boldsymbol{\tau} = \boldsymbol{\tau} + \boldsymbol{B}_p\boldsymbol{p}, \;\; \boldsymbol{Q}\boldsymbol{\nu} = \boldsymbol{\nu} + \boldsymbol{B}_q\boldsymbol{q}, \tag{8}$$

where

$$\begin{aligned} \boldsymbol{p} &= \left[p^{(12)}, \ldots, p^{(1M)}, \ldots, p^{(M-1,M)}\right]^T, \\ \boldsymbol{q} &= \left[q^{(12)}, \ldots, q^{(1M)}, \ldots, q^{(M-1,M)}\right]^T. \end{aligned}$$

Relaxation Variables

Define

$$\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K]\,, \;\; \boldsymbol{y} = \left[\boldsymbol{t}^T, \boldsymbol{f}^T, \boldsymbol{p}^T, \boldsymbol{q}^T\right]^T,$$

the cross terms

$$\begin{aligned}
&\boldsymbol{T} = \boldsymbol{t}\boldsymbol{t}^T,\; \boldsymbol{U} = \boldsymbol{t}\boldsymbol{f}^T,\; \boldsymbol{T}_p = \boldsymbol{t}\boldsymbol{p}^T,\; \boldsymbol{T}_q = \boldsymbol{t}\boldsymbol{q}^T,\\
&\boldsymbol{F} = \boldsymbol{f}\boldsymbol{f}^T,\; \boldsymbol{F}_p = \boldsymbol{f}\boldsymbol{p}^T,\; \boldsymbol{F}_q = \boldsymbol{f}\boldsymbol{q}^T,\\
&\tilde{\boldsymbol{P}} = \boldsymbol{p}\boldsymbol{p}^T,\; \tilde{\boldsymbol{Q}} = \boldsymbol{q}\boldsymbol{q}^T,\; \tilde{\boldsymbol{P}}_q = \boldsymbol{p}\boldsymbol{q}^T,
\end{aligned} \tag{9}$$

and the corresponding Gram matrices

$$\boldsymbol{Z} = \boldsymbol{X}^T\boldsymbol{X}, \tag{10}$$

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{T}_f & \boldsymbol{T}_p & \boldsymbol{T}_q \\ \boldsymbol{T}_f^T & \boldsymbol{F} & \boldsymbol{F}_p & \boldsymbol{F}_q \\ \boldsymbol{T}_p^T & \boldsymbol{F}_p^T & \tilde{\boldsymbol{P}} & \tilde{\boldsymbol{P}}_q \\ \boldsymbol{T}_q^T & \boldsymbol{F}_q^T & \tilde{\boldsymbol{P}}_q^T & \tilde{\boldsymbol{Q}} \end{bmatrix}, \;\; \bar{\boldsymbol{Y}} = \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{y} \\ \boldsymbol{y}^T & 1 \end{bmatrix}. \tag{11}$$

The Objective Function in a Simple Form

The objective function of (5) can be written in a simple form:

$$\theta = \mathrm{tr}(\bar{\boldsymbol{E}}\bar{\boldsymbol{Y}})$$

where

$$\bar{\boldsymbol{E}} = \begin{bmatrix} \bar{\boldsymbol{A}}_1 & \mathbf{0} & \boldsymbol{B}_t & \mathbf{0} & \bar{\boldsymbol{b}}_1 \\ \mathbf{0} & \bar{\boldsymbol{A}}_2 & \mathbf{0} & \boldsymbol{B}_f & \bar{\boldsymbol{b}}_2 \\ \boldsymbol{B}_t^T & \mathbf{0} & \bar{\boldsymbol{B}}_p & \mathbf{0} & \bar{\boldsymbol{d}}_1 \\ \mathbf{0} & \boldsymbol{B}_f^T & \mathbf{0} & \bar{\boldsymbol{B}}_q & \bar{\boldsymbol{d}}_2 \\ \bar{\boldsymbol{b}}_1^T & \bar{\boldsymbol{b}}_2^T & \bar{\boldsymbol{d}}_1^T & \bar{\boldsymbol{d}}_2^T & \bar{C}_1 + \bar{C}_2 \end{bmatrix}$$

and

$$\bar{\boldsymbol{A}}_1 = \bar{\boldsymbol{G}}^T \bar{\boldsymbol{G}}/\sigma_T^2,\ \bar{\boldsymbol{b}}_1 = -\bar{\boldsymbol{G}}^T \boldsymbol{\tau}/\sigma_T^2,\ \bar{\boldsymbol{d}}_1 = \boldsymbol{B}_p^T \boldsymbol{\tau}/\sigma_T^2,$$

$$\bar{C}_1 = \|\boldsymbol{\tau}\|^2/\sigma_T^2,\ \boldsymbol{B}_t = \bar{\boldsymbol{G}}^T \boldsymbol{B}_p/\sigma_T^2,\ \bar{\boldsymbol{B}}_p = \boldsymbol{B}_p^T \boldsymbol{B}_p/\sigma_T^2,$$

$$\bar{\boldsymbol{A}}_2 = \bar{\boldsymbol{G}}^T \bar{\boldsymbol{G}}/\sigma_F^2,\ \bar{\boldsymbol{b}}_2 = -\bar{\boldsymbol{G}}^T \boldsymbol{\nu}/\sigma_F^2,\ \bar{\boldsymbol{d}}_2 = \boldsymbol{B}_q^T \boldsymbol{\nu}/\sigma_F^2,$$

$$\bar{C}_2 = \|\boldsymbol{\nu}\|^2/\sigma_F^2,\ \boldsymbol{B}_f = \bar{\boldsymbol{G}}^T \boldsymbol{B}_q/\sigma_F^2,\ \bar{\boldsymbol{B}}_q = \boldsymbol{B}_q^T \boldsymbol{B}_q/\sigma_F^2.$$

Important Relations

Denote by $\boldsymbol{T}^{(kl)}$, $\boldsymbol{F}^{(kl)}$, and $\boldsymbol{U}^{(kl)}$ the (k,l)th block of $\boldsymbol{T}$, $\boldsymbol{F}$, and $\boldsymbol{U}$, respectively, where $k, l \in \mathcal{K}$. We have

$$\begin{aligned} T_{ii}^{(kk)} &= (t_i^{(k)})^2 = \frac{1}{c^2}\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|^2 \\ &= \frac{1}{c^2}\begin{bmatrix} \boldsymbol{s}_i \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{x}_k \\ \boldsymbol{x}_k^T & Z_{kk} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_i \\ -1 \end{bmatrix}, \end{aligned} \tag{12}$$

$$U_{ii}^{(kk)} = f_i^{(k)} t_i^{(k)} = \frac{1}{c^2}(\boldsymbol{s}_i - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_i, \tag{13}$$

$$\begin{aligned} T_{ij}^{(kl)} &= t_i^{(k)} t_j^{(l)} = \frac{1}{c^2}\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|\left\|\boldsymbol{x}_l - \boldsymbol{s}_j\right\| \\ &\geq \frac{1}{c^2}\left|(\boldsymbol{x}_k - \boldsymbol{s}_i)^T(\boldsymbol{x}_l - \boldsymbol{s}_j)\right| \quad \text{(by the Cauchy-Schwarz inequality)} \\ &= \frac{1}{c^2}\left| \begin{bmatrix} \boldsymbol{s}_i \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{x}_l \\ \boldsymbol{x}_k^T & Z_{kl} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_j \\ -1 \end{bmatrix} \right|, \end{aligned} \tag{14}$$

$$\begin{aligned} & T_{ij}^{(kl)} + F_{ij}^{(kl)} \pm U_{ij}^{(kl)} \pm U_{ji}^{(lk)} \\ & \leq \left(T_{ii}^{(kk)} + F_{ii}^{(kk)} \pm 2U_{ii}^{(kk)} + T_{jj}^{(ll)} + F_{jj}^{(ll)} \pm 2U_{jj}^{(ll)} \right) /2. \end{aligned} \tag{15}$$

We have the following bounds:

$$|f_i^{(k)}| \leq \|\dot{\boldsymbol{s}}_i\|/c, \tag{16}$$

$$U_{ij}^{(kl)} \leq \left(T_{ii}^{(kk)} + F_{jj}^{(ll)} \right) /2, \tag{17}$$

$$F_{ii}^{(kk)} \leq \|\dot{\boldsymbol{s}}_i\|^2/c^2, \tag{18}$$

$$\left| F_{ij}^{(kl)} \right| \leq \|\dot{\boldsymbol{s}}_i\|\|\dot{\boldsymbol{s}}_j\|/c^2. \tag{19}$$

According to the orthogonality between the columns of each permutation matrix, we have

$$(p^{(ij)})^2 = p^{(ij)}, \quad (q^{(ij)})^2 = q^{(ij)}.$$

This leads to

$$\text{diag}(\tilde{\boldsymbol{P}}) = \boldsymbol{p}, \quad \text{diag}(\tilde{\boldsymbol{Q}}) = \boldsymbol{q}. \tag{20}$$

Relaxations

With the above preparations, we can relax the constraints:

$$\boldsymbol{Z} = \boldsymbol{X}^T\boldsymbol{X} \;\rightarrow\; \boldsymbol{Z} \succeq \boldsymbol{X}^T\boldsymbol{X} \;\Longleftrightarrow\; \begin{bmatrix} \boldsymbol{I} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Z} \end{bmatrix} \succeq \boldsymbol{0}, \tag{21}$$

$$\boldsymbol{Y} = \boldsymbol{y}\boldsymbol{y}^T \;\rightarrow\; \boldsymbol{Y} \succeq \boldsymbol{y}\boldsymbol{y}^T \;\Longleftrightarrow\; \bar{\boldsymbol{Y}} \succeq \boldsymbol{0}, \tag{22}$$

and

$$\begin{aligned} &\boldsymbol{T} + \boldsymbol{F} \pm \boldsymbol{U} \pm \boldsymbol{U}^T = (\boldsymbol{t} \pm \boldsymbol{f})(\boldsymbol{t} \pm \boldsymbol{f})^T \\ \rightarrow\; &\boldsymbol{T} + \boldsymbol{F} \pm \boldsymbol{U} \pm \boldsymbol{U}^T \succeq (\boldsymbol{t} \pm \boldsymbol{f})(\boldsymbol{t} \pm \boldsymbol{f})^T \\ \Longleftrightarrow\; &\bar{\boldsymbol{Z}} = \begin{bmatrix} \boldsymbol{T} + \boldsymbol{F} \pm \boldsymbol{U} \pm \boldsymbol{U}^T & \boldsymbol{t} \pm \boldsymbol{f} \\ (\boldsymbol{t} \pm \boldsymbol{f})^T & 1 \end{bmatrix} \succeq \boldsymbol{0}. \end{aligned} \tag{23}$$

Semidefinite Relaxation

By putting all the pieces together, we finally arrive at the following SDR of Problem (3) when there are two unknown emitters:

$$\begin{aligned} \min \quad & \text{tr}(\bar{\boldsymbol{E}}\bar{\boldsymbol{Y}}) + \delta_1 \sum_{\substack{i,j\in\mathcal{I} \\ k,l\in\mathcal{K}}} T_{ij}^{(kl)} + \delta_2 \sum_{\substack{i,j\in\mathcal{I} \\ k,l\in\mathcal{K}}} \left| F_{ij}^{(kl)} \right| \\ \text{s.t.} \quad & \text{(12)–(23) satisfied.} \end{aligned} \tag{24}$$

Here, $\delta_1, \delta_2 \geq 0$ are penalty parameters used to induce a good solution to the original problem (3).

Geolocation Algorithm

Step 1: Choose a pair (δ_1, δ_2) $(\delta_i \in [10^{-6}, 10^{-1}])$*. Use solver SeDuMi or SDPT3 in CVX [20] to solve the SDR (24) and obtain the location estimates* $\hat{\boldsymbol{X}}$ *of the two unknown emitters and the corresponding* $\boldsymbol{p}$ *and* $\boldsymbol{q}$*.*

Step 2: Apply any local optimization routine (e.g., Newton-type methods) to the objective function of Problem (3) using $(\hat{\boldsymbol{X}}, \boldsymbol{p}, \boldsymbol{q})$ *as the initial point.*

Step 3: Based on the result in Step 2, perform a minimum weight perfect bipartite matching as outlined above to obtain permutation matrices $\{(\hat{\boldsymbol{P}}^{(ij)}, \hat{\boldsymbol{Q}}^{(ij)}) : i, j \in \mathcal{I},\ i > j\}$*.*

Step 4: Fix the permutation matrices obtained in Step 3. Then, Problem (3) becomes a standard geolocation problem using joint TDOA and FDOA measurements, which can be solved by standard SDR techniques. The location estimates thus obtained can then be further refined by local search.

[20] M. Grant and S. Boyd, CVX: Matlab Software for Disciplined Convex Programming, version {http://cvxr.com/cvx}, Jan. 2011.

Simulation Results

The true locations $(\times 10^6\ m)$ and velocities $(\times 10^3\ m/s)$ of three formation-flying satellites are obtained according to the routine in the Satellite Tool Kit (STK):

$$\boldsymbol{S} = \begin{bmatrix} -2.59693665 & -2.70482425 & -2.49791532 \\ 3.23460820 & 3.19406424 & 3.29006186 \\ 5.60250575 & 5.57715056 & 5.60777468 \end{bmatrix},$$

$$\dot{\boldsymbol{S}} = \begin{bmatrix} -7.017428 & -6.968662 & -7.061741 \\ -1.408503 & -1.457512 & -1.372917 \\ -2.439598 & -2.544959 & -2.340081 \end{bmatrix}.$$

The locations of the two unknown emitters are randomly chosen on the surface of the Earth, such as

$$\boldsymbol{x}_1 = [-2.53242580, 3.19230238, 4.91571459]^T \times 10^6 m,$$

$$\boldsymbol{x}_2 = [-2.53756142, 3.01653500, 5.02035731]^T \times 10^6 m,$$

which are within the coverage of a satellite formation (here we use S-type formation).

Simulation Results

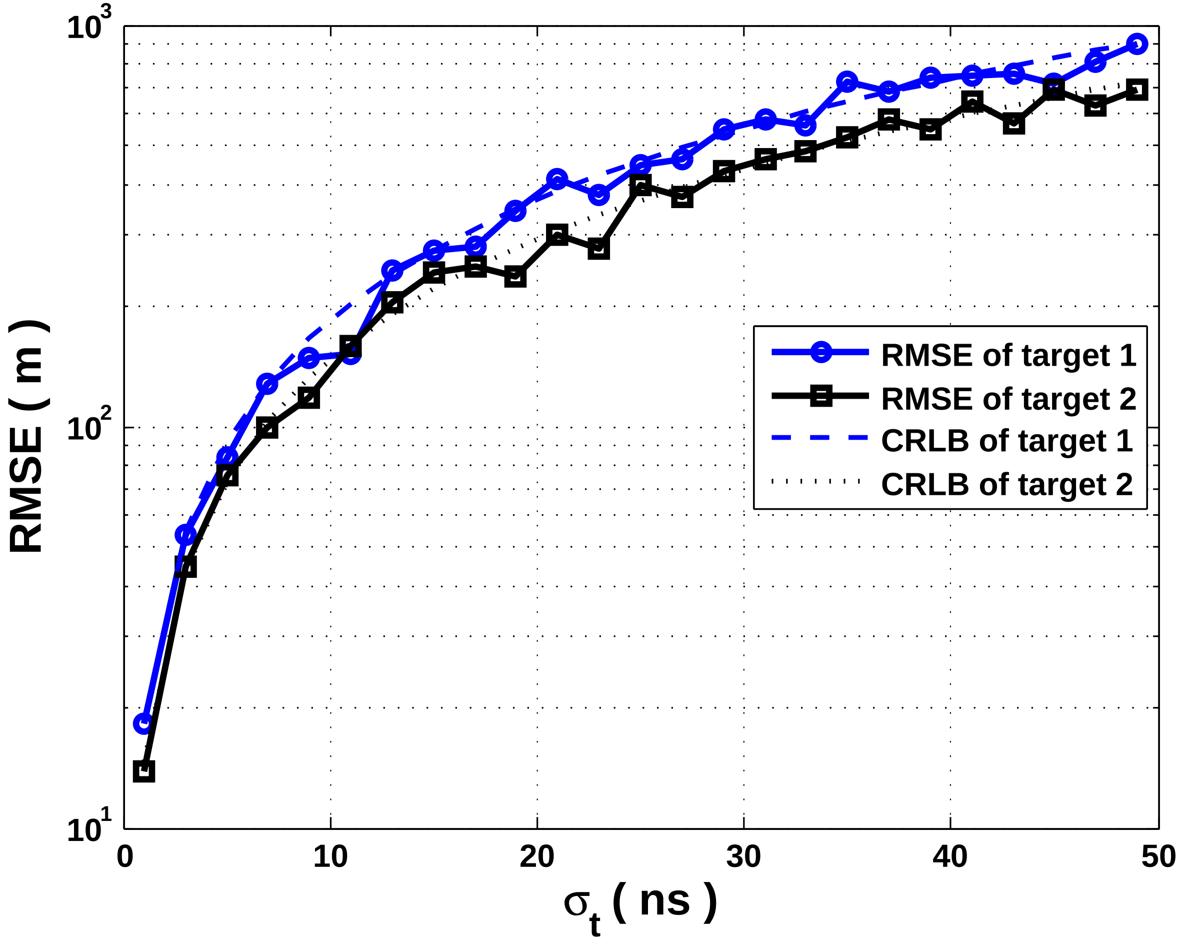


Figure 3: RMSE of Two Co-Channel Emitter Geolocation with $\sigma_f = 0.1\sigma_t$

Thank You !